

A Cubic Surface of Revolution

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Abstract

We develop a direct and elementary (calculus-free) exposition of the famous cubic surface of revolution $x^3 + y^3 + z^3 - 3xyz = 1$.

1 Introduction

A well-known exercise in classical differential geometry [1, 6, 16] is to show that the set \mathcal{S} of all points $(x, y, z) \in \mathbb{R}^3$ which satisfy the cubic equation

$$\boxed{F(x, y, z) \equiv x^3 + y^3 + z^3 - 3xyz - 1 = 0} \quad (1)$$

is a *surface of revolution*.

The standard proof ([6] and [16, p. 11]), which, in principle, goes back to LAGRANGE [9] and MONGE [13], is to verify that (1) satisfies the partial differential equation (here written as a determinant):

$$\begin{vmatrix} F_x(x, y, z) & F_y(x, y, z) & F_z(x, y, z) \\ x - a & y - b & z - c \\ l & m & n \end{vmatrix} = 0$$

which characterizes any surface of revolution $F(x, y, z) = 0$ whose axis of revolution has direction numbers (l, m, n) and goes through the point (a, b, c) . This PDE, for its part, expresses the geometric property that the normal line through any point of \mathcal{S} must intersect the axis of revolution (this is rather subtle; see [8]). All of this, though perfectly correct, seems complicated and rather sophisticated just to show that one can obtain \mathcal{S} by rotating a suitable curve around a certain fixed line. Moreover, to carry out this proof one needs to know *a priori* just what this axis is, something not immediately clear from the statement of the problem. Nor does the solution give much of a clue as to *which* curve one rotates.

A search of the literature failed to turn up a treatment of the problem which differs significantly from that sketched above (although see [1]).

The polynomial (1) is quite famous and has been the object of numerous algebraical and number theoretical investigations. See the delightful and informative paper [12]. See also the remarks at the end of this paper. It therefore is all the more surprising that an elementary treatment of its geometrical nature as a surface of revolution is apparently not to be found in any readily available source.

Therefore, this paper offers two detailed fully elementary and calculus-free solutions of the problem based on simple coordinate geometry. We will obtain a parametric representation of a meridian curve whose rotation produces \mathcal{S} as well as a parametric representation of \mathcal{S} itself, which we have not seen before (although it can hardly be new).

Moreover, we relate the surface \mathcal{S} to the general theory of cubic surfaces, of which there is an enormous literature (see [10]), and in particular we prove a version of the famous Salmon–Cayley theorem which asserts that any nonsingular (complex) cubic surface contains 27 straight lines.

Finally, this parametrization of \mathcal{S} and the theory of *pythagorean triples* will permit us to find infinitely many *rational* points on the surface \mathcal{S} defined by the equation (1) via our rational parametrization of \mathcal{S} .

2 The first elementary solution

The celebrated factorization

$$x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \quad (2)$$

is an endless source of Olympiad problems and is the basis of our first solution. Let

$$t := x + y + z, \quad (3)$$

where we assume $t > 0$. Indeed, since the second factor in (2) is

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}(x - y)^2 + \frac{1}{2}(x - z)^2 + \frac{1}{2}(y - z)^2,$$

and since we are only interested in finite real points, it cannot be negative, so *there is no solution to (1) with $t \leq 0$* .

Now we write the equation of \mathcal{S} in the form

$$x^2 + y^2 + z^2 = \frac{2}{3(x + y + z)} + \frac{(x + y + z)^2}{3} \equiv \frac{2}{3t} + \frac{t^2}{3} \quad (4)$$

which is legitimate because we just showed that $x + y + z \neq 0$ on \mathcal{S} . This is the equation of a sphere whose center is at the origin and whose radius is $\sqrt{\frac{2}{3t} + \frac{t^2}{3}}$.

Let Σ_t denote this sphere (4) and let Π_t denote the plane $x + y + z = t$. Finally, let

$$\Gamma_t := \Sigma_t \cap \Pi_t.$$

We can see that Γ_t is nonempty for $t > 0$ since the square of the distance from the origin to Π_t is $t^2/3$ and this is less than the square of the radius $2/3t + t^2/3$ of the sphere Σ_t .

Thus Γ_t is a circle with center on the line $x = y = z$ and in the plane Π_t orthogonal to that line. If the intersection were a single point, the distance from the origin to the plane would equal the radius of the sphere, that is, $t^2/3 = 2/3t + t^2/3$, which is impossible. Then it follows that $\Gamma_t \subset \mathcal{S}$ for all $t > 0$ and that, therefore

$$\mathcal{S} = \bigcup_{t>0} \Gamma_t.$$

Moreover, the pythagorean theorem now shows that the square of the radius of Γ_t is $2/3t$. Therefore, we have proved the following result.

Theorem 1. *The surface \mathcal{S} is a **surface of revolution** formed by the union of all the circles with variable center at $(\frac{t}{3}, \frac{t}{3}, \frac{t}{3})$, $0 < t < \infty$, and radius $\sqrt{2/3t}$. Each such circle lies in the plane $x + y + z = t$, which cuts the corresponding line $x = y = z$ perpendicularly. \square*

We add the remark that the equation

$$x^3 + y^3 + z^3 - r \cdot xyz = 1,$$

where $r \in \mathbb{R}$, is a surface of revolution *only* for $r = 3$. (We prove this later on.) Thus, our equation (1) is quite special.

3 Parametrizations

Now we can parametrize a meridian curve of \mathcal{S} . We recall that the intersection with \mathcal{S} of any plane that contains the axis of revolution of \mathcal{S} is a meridian curve of \mathcal{S} . The plane $2z = x + y$ contains the line $x = y = z$ and its normal has direction numbers $(1, 1, -2)$. A unit vector parallel to this normal is $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$. Therefore the vector

$$\mathbf{r}_1 := \sqrt{\frac{2}{3t}} \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right)$$

is a normal vector to $x = y = z$ and its length is the radius of the circle of intersection. Forming its vector sum with $(\frac{t}{3}, \frac{t}{3}, \frac{t}{3})$, we have proved the following result.

Theorem 2. *The following parameterization gives us a meridian curve $\mathbf{C}(t)$ of \mathcal{S} :*

$$\mathbf{C}(t) = \left(\frac{t}{3} + \frac{1}{3\sqrt{t}}, \frac{t}{3} + \frac{1}{3\sqrt{t}}, \frac{t}{3} - \frac{2}{3\sqrt{t}} \right)$$

where $0 < t < \infty$. \square

A vector perpendicular to $x = y = z$ and to $(1, 1, -2)$ simultaneously is $(1, -1, 0)$. A unit vector parallel to it is $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$ and

$$\mathbf{r}_2 := \sqrt{\frac{2}{3t}} \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right)$$

is a normal vector to $x = y = z$ whose length is the radius of the circle of intersection.

Moreover, as we noted earlier, $\mathbf{r}_1 \perp \mathbf{r}_2$.

Therefore, we can parameterize the surface \mathcal{S} as follows:

$$\mathbf{r}(t, \theta) := \left(\frac{t}{3}, \frac{t}{3}, \frac{t}{3} \right) + \mathbf{r}_1 \cos \theta + \mathbf{r}_2 \sin \theta$$

or, writing $\mathbf{r}(t, \theta) := (x(t, \theta), y(t, \theta), z(t, \theta))$, we have proved the following result.

Theorem 3. *A coordinate parametrization of \mathcal{S} is*

$$\begin{aligned} x(t, \theta) &= \frac{t}{3} + \frac{1}{3\sqrt{t}} \cos \theta + \frac{1}{\sqrt{3t}} \sin \theta \\ y(t, \theta) &= \frac{t}{3} + \frac{1}{3\sqrt{t}} \cos \theta - \frac{1}{\sqrt{3t}} \sin \theta \\ z(t, \theta) &= \frac{t}{3} - \frac{2}{3\sqrt{t}} \cos \theta \end{aligned}$$

where $0 < t < \infty$ and $0 \leq \theta < 2\pi$. □

It is a pleasant surprise that the cubic surface \mathcal{S} has an elementary parametrization. However, we could have predicted the existence of such a parameterization *a priori*. For, it is shown in the general theory of cubic surfaces [15] that a real cubic surface has a rational parametrization over the real numbers if and only if its real support is a connected set. However, it is not easy to find such parameterizations and much research has been dedicated to creating algorithms for producing them (again, see [15]). We conjecture that the real surface represented by the equation

$$x^3 + y^3 + z^3 - r \cdot xyz = 1$$

where $r \in \mathbb{R}$, is connected if and only if $-\infty < r \leq 3$. This conjecture seems difficult to prove for $r < 3$ although it is evident geometrically if one uses computer generated graphs for suitable values of r . Moreover, the theorem on the existence of parameterizations for connected cubic surfaces requires more advanced techniques than are appropriate for our paper. We add that it is well known that the trigonometric functions in our parametrization can be replaced by suitable rational functions of a single parameter. Indeed, we do so in Section 7 below.

4 An alternate solution

Another way to show that \mathcal{S} is a surface of revolution is to rotate the surface \mathcal{S} around the origin in such a way that the plane $x + y + z = 0$ becomes the new XY -plane, the line $x = y = z$ becomes the new Z -axis and the line $x + y = 0, z = 0$ becomes the new X -axis. This is accomplished by the rotation equations:

$$\begin{aligned} x &= \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{6}} + \frac{Z}{\sqrt{3}}, \\ y &= -\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{6}} + \frac{Z}{\sqrt{3}}, \\ z &= -\frac{2Y}{\sqrt{6}} + \frac{Z}{\sqrt{3}}. \end{aligned}$$

We find that the surface \mathcal{S} has the equation

$$Z = \frac{2}{3\sqrt{3}(X^2 + Y^2)}$$

which explicitly shows that it is a surface of revolution around the Z -axis obtained by rotating the curve $Z = \frac{2}{3\sqrt{3}Y^2}$.

We remark that the same rotation equations, applied to the equation

$$x^3 + y^3 + z^3 - r \cdot xyz = 1$$

where $r \in \mathbb{R}$, gives an equation which is of the form $Z = f(X^2 + Y^2)$ if and only if $r = 3$, and therefore *is not a surface of revolution* with the axis $x = y = z$ for $r \neq 3$.

5 Singular points of cubic surfaces

Let $F(w, x, y, z)$ be an irreducible complex homogeneous polynomial of (total) degree three in the polynomial ring $\mathbb{C}[w, x, y, z]$. Then the point $(w, x, y, z) = (w_0, x_0, y_0, z_0) \equiv \mathbf{p}$ is a *singular point* of the algebraic surface \mathbf{S} defined by the equation $F(w, x, y, z) = 0$ if and only if

$$\nabla F(\mathbf{p}) = \mathbf{0};$$

that is, if and only if

$$F(\mathbf{p}) = F_w(\mathbf{p}) = F_x(\mathbf{p}) = F_y(\mathbf{p}) = F_z(\mathbf{p}) = 0.$$

The surface \mathbf{S} is called *singular* if it has a singular point, otherwise it is called *nonsingular*.

Our cubic surface, \mathcal{S} , turns out to be singular. Indeed, the singular points defined by

$$x^3 + y^3 + z^3 - 3xyz = w^3 \tag{5}$$

are

$$(0, 1, 1, 1), \quad (0, 1, \varepsilon, \varepsilon^2), \quad (0, 1, \varepsilon^2, \varepsilon),$$

where ε is a complex cube root of unity.

The singular points of the rotated form

$$z(x^2 + y^2) - \frac{2}{3\sqrt{3}}w^3 = 0 \quad (6)$$

are

$$(0, i, 1, 0), \quad (0, -i, 1, 0), \quad (0, 0, 0, 1).$$

Singular points on a cubic surface can be grouped into different classes [5, p. 135].

Suppose that $(w, x, y, z) = (w_0, x_0, y_0, z_0)$ is an isolated singular point and that the Taylor expansion is:

$$\begin{aligned} F(w_0, x_0 + x, y_0 + y, z_0 + z) = & \alpha_{11}x^2 + \alpha_{22}y^2 + \alpha_{33}z^2 \\ & + 2\alpha_{12}xy + 2\alpha_{13}xz + 2\alpha_{23}yz + F_3(x, y, z), \end{aligned}$$

where $F_3(x, y, z)$ is homogeneous of degree 3. The matrix of the coefficients of the above quadratic form is

$$\alpha := \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}.$$

If the determinant of α is zero (i.e., the matrix is singular) and if the matrix has rank 2, then the singular point is a *biplanar double point* or a *binode*. The term originates from the surface having two tangent planes at such a point. For example, the surface defined by the equation $z^3 - x^2 + y^2 = 0$ has a binode at the origin with tangent planes $x \pm y = 0$.

In order to apply these definitions to the surface, we make the affine change of variable

$$X := x + iy, \quad Y := x - iy, \quad Z := z, \quad W := -\left(\frac{3\sqrt{3}}{2}\right)^{1/3} w$$

in the rotated equation. Then the equation of the surface (6) becomes

$$XYZ + W^3 = 0, \quad (7)$$

and its singularities are

$$(0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1).$$

Then, at the point $\mathbf{p} = (w_0, x_0, y_0, z_0)$ our matrix is

$$\alpha = \begin{pmatrix} 0 & z_0 & y_0 \\ z_0 & 0 & y_0 \\ y_0 & x_0 & 0 \end{pmatrix}.$$

This shows that *each of our singular points is a binode in the plane $W = 0$.*

6 The twenty seven lines on a cubic surface

In 1849, SALMON [16, p. 183] proved the celebrated theorem that every nonsingular cubic surface contains twenty seven (real and/or complex) straight lines. Complete treatises (see [11]) have been written on this theorem and it continues to be a source of modern research [10]. Sixteen years later, SCHLÄFLI [17], and six years after that, CAYLEY [3] wrote long papers, not altogether easy to read, extending Salmon's theorem to *singular* cubic surfaces. They classified cubic surfaces into 23 “species” according to the kind of singularities they possess, and found the number of lines associated with each species. The presence of singularities *decreases* the number of lines, and the smallest number on a singular surface is 3. Then, more than a century later (1978), BRUCE and WALL [2] revisited the work of Schläfli and Cayley using modern techniques and obtained 21 species of cubic surfaces.

Since our cubic surface \mathcal{S} has three binodes as singularities, Schläfli [17, p. 239] and Cayley, [3, pp. 320–321] placed it in class XXI while Bruce and Wall [2, p. 253] placed it in class $3A_2$. Both classifications assign *three lines* to the surface. If we refer to equation (7), the three lines are

$$X = 0, W = 0; \quad Y = 0, W = 0; \quad Z = 0, W = 0;$$

while the lines for the original cubic (5) are

$$x + y + z = 0, w = 0; \quad x + \varepsilon y + \varepsilon^2 z = 0, w = 0; \quad x + \varepsilon^2 y + \varepsilon z = 0, w = 0;$$

where ε is a complex cubic root of unity. These are all lines “at infinity” since $w = 0$ for all three. Moreover, the first line is real and the other two are complex. According to Schläfli and Cayley, the first line belongs to subspecies XXI.1 and the other two “conjugate” lines (in their terminology) belong to subspecies XXI.2.

The proof that these are the *only* lines belonging to our surface is a consequence of the general theories these authors develop: but that proof is not easily distilled to our special case.

Thus, it is of interest to directly prove this theorem for our own special cubic surface \mathcal{S} . We will first prove the following special case.

Theorem 4. *The cubic surface of revolution \mathcal{S} does not contain any finite real lines.*

Proof. The rotated form of the equation for \mathcal{S} shows that $Z > 0$ for all (finite) points of the surface. Therefore, any line wholly contained in \mathcal{S} cannot intersect the XY -plane, i.e., it must be *parallel* to the plane $Z = 0$. Any such line has the parametric representation

$$\ell = (at + b, ct + d, e),$$

where $t \in \mathbb{R}$ while a, b, c, d and $e > 0$ are real constants. Since ℓ is wholly contained in \mathcal{S} the three coordinates in the parametric representation identically satisfy the equation for \mathcal{S} :

$$e[(at + b)^2 + (ct + d)^2] - \frac{2}{3\sqrt{3}} = 0.$$

If we divide by e and rearrange this in powers of t , we find that the coefficient of t^2 is

$$a^2 + c^2.$$

Since this must hold for all real t , the individual coefficients must all vanish. In particular, the equation

$$a^2 + c^2 = 0$$

must hold. This means that

$$a = c = 0,$$

which in turn means that the line is the point (b, d, e) . This contradiction shows that any such line in \mathcal{S} must be nonreal. \square

We thank the referee for his elegant proof of the following more general result.

Theorem 5. *The cubic surface of revolution \mathcal{S} does not contain any finite real or complex lines.*

Proof. Take $W = -1$ in equation (7) and write the coordinates as (x, y, z) instead of (X, Y, Z) . Therefore, our equation is

$$xyz = 1. \tag{8}$$

Any line \mathcal{L} in \mathcal{S} other than the three at infinity must be the join of two distinct finite points of \mathcal{S} , say $(a, b, c) \in \mathbb{C}^3$ and $(A, B, C) \in \mathbb{C}^3$. The general point of \mathcal{L} is given by

$$(1 - \lambda)(a, b, c) + \lambda(A, B, C), \quad \text{where } \lambda \in \mathbb{C}.$$

Therefore, equation (8) becomes

$$[\lambda a + (1 - \lambda)A] \cdot [\lambda b + (1 - \lambda)B] \cdot [\lambda c + (1 - \lambda)C] = 1, \tag{9}$$

and this must hold for all $\lambda \in \mathbb{C}$. Dividing (9) by λ^3 and letting $\lambda \rightarrow \infty$, we obtain

$$(a - A)(b - B)(c - C) = 0.$$

Therefore either $a = A$, or $b = B$, or $c = C$.

Suppose $a = A$. Then equation (9) becomes

$$a \cdot [\lambda b + (1 - \lambda)B] \cdot [\lambda c + (1 - \lambda)C] = 1 \tag{10}$$

for all $\lambda \in \mathbb{C}$. This shows, in particular, that $a \neq 0$. Divide (10) by λ^2 and let $\lambda \rightarrow \infty$. Then (10) becomes

$$a(b - B)(c - C) = 0.$$

But $a \neq 0$, which means that either $b = B$ or $c = C$, necessarily. But either alternative implies the other, which means that

$$(a, b, c) = (A, B, C),$$

and the line \mathcal{L} collapses to a single point. This contradicts the assumption that $(a, b, c) \in \mathbb{C}^3$ and $(A, B, C) \in \mathbb{C}^3$ are distinct finite points.

Finally, the alternatives $b = B$ or $c = C$ lead to the same false conclusion, and therefore there are NO finite lines in \mathcal{S} . \square

We note that both proofs, though based on totally different ideas, lead to the same contradiction, namely that the line, supposed to exist, collapses to a single point.

7 Rational points on a cubic surface

The history of the celebrated problem of finding rational points on a cubic surface is detailed in Chapter XXI of Dickson's monumental work on the history of the theory of numbers [7]. Subsequently the great british mathematician L.J. MORDELL made fundamental contributions to solving this problem and he summarized them in his classic book [14]. In particular, on page 82 he states:

“No method is known for determining whether rational points exist on a general cubic surface $f(x, y, z) = 0$, or for finding all of them if any exist. Geometric considerations may prove very helpful and sometimes by their help an infinity of solutions may be found...”

This statement continues to be true, today. And, as we will see, “geometric considerations” will lead us to an infinity of rational points on \mathcal{S} .

If we think of the equation (1) as an equation in the three unknowns (x, y, z) , we can obtain rational solutions by taking

$$t =: u^2, \quad \sin \theta =: \frac{2r\sqrt{3}}{r^2 + 3}, \quad \cos \theta =: \frac{r^2 - 3}{r^2 + 3},$$

in the parametric representation of \mathcal{S} where u and r run over all rational numbers. Then we obtain:

Theorem 6. *If $u \neq 0$ and r run over all rational numbers then the following formulas*

$$\begin{aligned} x &= \frac{u^2}{3} + \frac{1}{3u} \frac{r^2 - 3}{r^2 + 3} + \frac{1}{u} \frac{2r}{r^2 + 3}, \\ y &= \frac{u^2}{3} + \frac{1}{3u} \frac{r^2 - 3}{r^2 + 3} - \frac{1}{u} \frac{2r}{r^2 + 3}, \\ z &= \frac{u^2}{3} - \frac{2}{3u} \frac{r^2 - 3}{r^2 + 3}, \end{aligned}$$

*furnish infinitely many **rational points** on the cubic surface defined by*

$$x^3 + y^3 + z^3 - 3 \cdot xyz = 1.$$

□

For example, if we take $u = 2$ and $r = \frac{1}{3}$, we obtain the solution $x = \frac{9}{7}, y = \frac{15}{14}, z = \frac{23}{14}$.

Our rather *ad hoc* formulas for $\cos \theta$ and $\sin \theta$ are based on the standard formulas for the *Pythagorean triples* as applied to the give the complete (positive) rational number solution to the equation $x^2 + y^2 = 1$. Namely, the solution to $a^2 + b^2 = c^2$ given by $a = 2mn, b = m^2 - n^2, c = m^2 + n^2$ $m > n$ where m and n run through all integers becomes $x = \frac{2r}{r^2+1}, y = \frac{r^2-1}{r^2+1}$ where $r = \frac{m}{n}, n \neq 0$ runs through all positive rational numbers and we take $x = \sin \theta, y = \cos \theta$. In order to cancel the term $\sqrt{3}$ in the denominator of the term multiplying $\sin \theta$ in the coordinate parametrization of \mathcal{S} we replace the numerator $2r$ by $\sqrt{3} \cdot 2r$. But, in order to maintain the identity $x^2 + y^2 = 1$ the constant $+1$ in there formulas for x and y must be replaced by $+3$ and our rational number parametrization results.

Although our rational parametrization of \mathcal{S} gives infinitely many rational solutions to the cubic equation (1), it does not give all rational solutions. For example, the solution

$$x = \frac{18}{7}, \quad y = \frac{16}{7}, \quad z = \frac{15}{7} \quad (11)$$

is not given by our formulas as the reader can check by eliminating u and using the rational root theorem on the sextic polynomial equation that results for r .

We only mention it to show that our parametric representation of \mathcal{S} , when conjoined with the famous formulas for Pythagorean triples, gives us a nice bonus in the form of a rational parametrization of \mathcal{S} . The complete rational solution, as well as references to the work of RAMANUJAN and others on this equation can be found in [4].

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